A class of nonlinear differential equations of second order in time

Michelle Schatzman

A CLASS OF NONLINEAR DIFFERENTIAL EQUATIONS OF SECOND ORDER IN TIME

MICHELLE SCHATZMAN
Analyse Numerique t55–65, Universite P. et M. Curie, 4pl Jussieu, 75230 Paris Cedex 05, France

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1. INTRODUCTION

IN WHAT follows, ϕ will denote a lower semicontinuous convex proper function from $\mathbf{R}^n = H$ to $\mathbf{R} \cup \{+\infty\}$. Here proper means that $\phi \not\equiv +\infty$. The effective domain of ϕ is the set dom $\phi = \{x \in \mathbf{R}^N/\phi(x) < +\infty\}$. We shall suppose that the interior of dom ϕ in $\mathbf{R}^N$ is not empty, and $\phi \geqslant 0$. These two assumptions do not restrict the generality. The scalar product in H is denoted by (x, y).

Let us recall that the subdifferential of ϕ is a (multivalued) operator defined on

$$D(\partial\phi) = \{x \in H : \exists\, z \in H \text{ such that } \phi(x + y) - \phi(x) \geqslant (z, y) \text{ for all } y \in H\}$$

with values

$$\partial\phi(x) = \{z \in H : \phi(x + y) - \phi(x) \geqslant (z, y), \forall y \in H\}.$$

The set $\partial\phi(x)$ is closed convex in H and $\partial\phi$ is a cyclically monotone operator, i.e. for all $n \in \mathbf{N}$, for all n-tuple $(x_1, \ldots, x_n)$ of elements of $D(\partial\phi)$, for all $y_i \in \partial\phi(x_i)$ $(1 \leqslant i \leqslant n)$

$$(y_n, x_n - x_{n-1}) + , \ldots + (y_2, x_2 - x_1) + (y_1, x_1 - x_n) \geqslant 0.$$

We define $J_\lambda = (I + \lambda\partial\phi)^{-1}$ on H; J_λ is a contraction, and $\partial\phi_\lambda = (I + J_\lambda)/\lambda$ is the Yosida approximation to $\partial\phi$; $\partial\phi_\lambda$ is Lipschitzian, with Lipschitz constant $1/\lambda$ and $\partial\phi_\lambda(x) \in \partial\phi(J_\lambda x)$. If $\phi_\lambda(x) = \inf\{|x - y|^2/(2\lambda) + (\phi/2)(y)\}$, $\partial\phi_\lambda$ is the subdifferential of ϕ_λ. Explicitly, $\phi_\lambda(x) = \phi(J_\lambda x) + |x + J_\lambda x|^2/(2\lambda)$ and $\phi_\lambda(x) \leqslant \phi(x)$. Moreover, $\lim_{\lambda \to 0} \phi_\lambda(x) - \phi(x)$ if $x \in H$. The general references for these results are refs [1, 2].

Before we proceed to the existence theorem, it is of some interest to give an explicit example. It will allow the reader to notice the difficulties of the problem.

Take $H = \mathbf{R}$, $K = \mathbf{R}^+$, $\phi = \psi$ the indicator function of K, i.e.

$$\phi(x) = +\infty \qquad \text{if } x \notin K$$

$$\phi(x) = 0 \qquad \text{if } x \in K$$

Then

$$\partial\phi(x) = \{0\} \qquad \text{if } x > 0$$

$$\partial\phi(x) = (-\infty; 0] \quad \text{if } x = 0$$

$$\partial\phi(x) = \varnothing \qquad \text{if } x < 0.$$

Assume $u_0 > 0$; we shall seek a solution u, which is locally Lipschitzian in t. The interesting case occurs when $u_1 < 0$. If $t < u_0/u_1$, the only solution of

$$\frac{d^2u}{dt^2} + \partial\psi_k(u) \ni 0$$

is $u(t) = x_0 + tu_1$. We must have $u(t) \geqslant 0$, for all $t \geqslant 0$. It is easy to check that any u of the form

$$u(t) = 0 \quad \text{for} \quad t_0 = -u_0/u_1 \leqslant t \leqslant t_1$$
$$u(t) = v(t - t_1) \quad \text{for} \quad t \geqslant t_1$$

with $v > 0$ and $-u_0/u_1 \leqslant t_1 \leqslant \infty$ is a solution. (see the figure below). If we assume that the energy is conserved, then necessarily $u^0(t) = -u_1(t + u_0|u_1)$ for $t \geqslant -u_0/u_1$. There is an

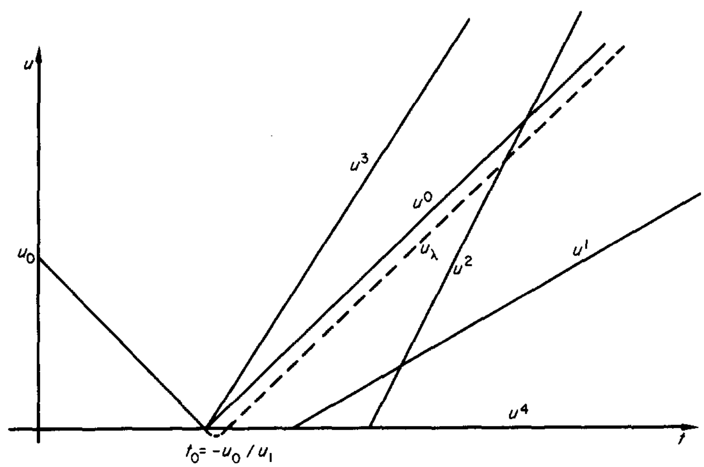

Fig. 1.

"optical reflection" in $-u_0/u_1$. The Yosida approximation gives, in the limit, the solution which conserves the energy. In fact, the approximating equation is

$$\frac{d^2u_\lambda}{dt^2} - \frac{u_\lambda^-}{\lambda} = 0, \text{ where } r^- = -\min(r, 0)\text{; its solution is}$$

$$u_\lambda(t) = u_0 + tu_1 \qquad 0 \leqslant t \leqslant -u_0/u_1 = t_0$$
$$u_\lambda(t) = \sqrt{\lambda}u_1 \sin\left[(t + t_0)/\sqrt{\lambda}\right] \qquad t_0 \leqslant t \leqslant t_0 + \pi\sqrt{\lambda}$$
$$u_\lambda(t) = -u_1(t - \pi\sqrt{\lambda} - t_0) \qquad t_0 + \pi\sqrt{\lambda} \leqslant t.$$

Clearly, the u_λ converge to the energy conserving solution, u^0.

2. THE EXISTENCE THEOREM

Definition 1. If $f \in L^2(0, T; H)$ (T finite) and $u_0 \in \operatorname{dom} \phi$, $u_1 \in H$, we say that u is an energy-conserving solution of the problem (P)

(P) $$(\mathrm{d}^2u/\mathrm{d}t^2) + \partial\phi(u) \ni f, \qquad \text{"initial conditions"}$$

when it satisfies the following requirements:

(1) $u \in W^{1,\infty}((0, T); H)$
(2) $u(t) \in \operatorname{dom}\phi, \forall t \in [0, T]$
(3) there exists a bounded measure μ with values in H, such that
$$(\mathrm{d}^2u/\mathrm{d}t^2) + \mu = f \text{ in the sense of distributions,}$$

and

(4) for any continuous v, with values in H, such that $\phi(v) \in L^1(0, T)$, we have:

$$\int_0^T (\phi(v) - \phi(u))\,\mathrm{d}t \geqslant \langle \mu, v - u\rangle$$

(5) $\mathrm{d}u/\mathrm{d}t$ has left and right limits for any $t \in [0, T]$ (with the necessary modifications at 0 and T)
(6) energy is conserved:

$$\left|\frac{\mathrm{d}^+u}{\mathrm{d}t}(t)\right|^2 + \phi(u(t)) = \left|\frac{\mathrm{d}^-u}{\mathrm{d}t}(t)\right|^2 + \phi(u(t)) = |u_1|^2 + \phi(u_0) + \int_0^t (\dot{u}(s), f(s))\,\mathrm{d}s$$

almost everywhere on $[0, T]$; here $\dot{u}$ stands for $\mathrm{d}u/\mathrm{d}t$
(7) the initial conditions are satisfied in the following sense:

$$u(0) = u_0$$

and if K_0 is the closure of dom ϕ, ψ_{K_0} the indicator function of K_0:

$$-u_1 + \frac{\mathrm{d}^+u}{\mathrm{d}t}(0) + \partial\psi_{K_0}(u_0) \ni 0.$$

THEOREM 1. (a) For any $f \in L^2(0, T; H)$, $u_0 \in \operatorname{dom}\phi$ and $u_1 \in H$, the problem (P) has an energy-conserving solution in the sense of Definition 1. This solution is obtained as the strong limit in $H^1((0, T); H)$ and the weak * limit in $W^{1,\infty}((0, T); H)$ of a subsequence of the sequence of solutions of (P_λ):

(P_λ) $$\frac{\mathrm{d}^2u_\lambda}{\mathrm{d}t^2} + \partial\phi_\lambda(u_\lambda) = 0, \qquad u_\lambda(0) = u_0, \qquad \frac{\mathrm{d}u_\lambda}{\mathrm{d}t}(0) = u_1.$$

(b) Moreover, if ϕ is Lipschitzian in a neighbourhood of u_0 with respect to K_0, and if $-u_1$ belongs to C, the tangent cone to K_0 at u_0 (i.e. $C = \overline{\bigcup_{\tau>0} \tau(K_0 - u_0)}$), then

(8) $$\frac{\mathrm{d}^+u}{\mathrm{d}t}(0) = 2P_C u_1 - u_1$$

where P_C is the projection onto C.

Proof. **Part (a)**

1. Preliminary estimates

We have the energy equality for (P_λ)

(9) $$\tfrac{1}{2}|\dot{u}_\lambda(t)|^2 + \phi_\lambda(u_\lambda(t)) = \tfrac{1}{2}|u_1|^2 + \phi_\lambda(u_0) + \int_0^t (\dot{u}_\lambda(s), f(s))\,\mathrm{d}s$$

from which follows the inequality

$$\tfrac{1}{2}|\dot{u}_\lambda(t)|^2 \leqslant \tfrac{1}{2}|u_1|^2 + \phi(u_0) + |f|_{L^2(0,T;H)}\left(\int_0^t |\dot{u}_\lambda(s)|^2\,\mathrm{d}s\right)^{\frac{1}{2}}$$

$$\leqslant \tfrac{1}{2}|u_1|^2 + \phi(u_0) + \tfrac{1}{2}|f|_{L^2(0,T;H)}\left(1 + \int_0^t |\dot{u}_\lambda(s)|^2\,\mathrm{d}s\right)$$

by the relation $x \leqslant (1 + x^2)/2$.
Gronwall's lemma implies:

$$\phi_\lambda(u_\lambda(t)) + |\tfrac{1}{2}\dot{u}_\lambda(t)|^2 \leqslant (\tfrac{1}{2}|u_1|^2 + \phi(u_0) + \tfrac{1}{2}|f|_{L^2(0,T;H)}) \exp(T|f|_{L^2(0,T;H)})$$

Set

(10) $$E^2/2 = (\tfrac{1}{2}|u_1|^2 + \phi(u_0) + \tfrac{1}{2}|f|_{L^2(0,T;H)}) \exp(T|f|_{L^2(0,T;H)})$$

We may therefore extract a subsequence, still denoted by u_λ, such that

$$u_\lambda \to u \text{ in } C^0([0,T]; H) \text{ strong}$$

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \rightharpoonup \frac{\mathrm{d}u}{\mathrm{d}t} \text{ in } L^\infty(0,T; H) \text{ weak*}$$

$$\phi_\lambda(u_\lambda) \rightharpoonup \chi \quad \text{in} \quad L^\infty(0,T) \text{ weak*}.$$

As $\phi_\lambda(u_\lambda) \leqslant E$, $|u_\lambda - J_\lambda u_\lambda| \leqslant \sqrt{2\lambda E}$ and $\phi(J_\lambda u_\lambda) \leqslant E$. We have

$$E \geqslant \lim_{\lambda \to 0} \phi(J_\lambda u_\lambda(t)) \geqslant \phi(u(t))$$

and $u(t) \in \operatorname{dom} \phi$, for all t in $[0,T]$. Hence we obtain (1) and (2).

2. Main estimate

Let a be the center of a closed ball of radius ρ, contained in the interior of dom ϕ. By general theorems, we may suppose that ϕ is bounded on $a + B_\rho$ by a constant C. For an arbitrary continuous z such that $|z(t)| \leqslant 1$ for all t in $[0,T]$, we may write

$$(\partial\phi_\lambda(u(t)), a + z(t) - u(t)) \leqslant \phi_\lambda(a + \rho z(t)) - \phi_\lambda(u_\lambda(t)).$$

Integrating from 0 to T, we have

$$\rho \int_0^T (\partial\phi_\lambda(u_\lambda(t)), z(t))\,\mathrm{d}t \leqslant CT + \int_0^T \left(\frac{\mathrm{d}^2 u_\lambda}{\mathrm{d}t^2} - f, a - u_\lambda\right)\mathrm{d}t \leqslant$$

$$\leqslant CT + \left(\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}, a - u_\lambda\right)\Big|_0^T + \int_0^T \left|\frac{\mathrm{d}u_\lambda}{\mathrm{d}t}\right|^2 \mathrm{d}t + \int_0^T (f, a - u_\lambda)\,\mathrm{d}t$$

$$\leqslant CT + E(2|a| + 2|u_0| + TE) + TE^2 + |f|_{L^2(0,T;H)}\sqrt{T}(|a| + |u_0| + TE).$$

If we choose

$$z(t) = \frac{1}{\rho}\frac{\partial\phi_\lambda(u_\lambda(t))}{|\partial\phi_\lambda(u_\lambda(t))|}$$

we obtain the estimate

(11)
$$\int_0^T |\partial\phi_\lambda(u_\lambda(t))|\,\mathrm{d}t \leqslant C' \text{ independent of } \lambda.$$

We may therefore extract a new subsequence, still denoted by u_λ, such that $\partial\phi_\lambda(u_\lambda) \to \mu$ vaguely in $M^1([0, T]; H)$, the set of bounded measures on $[0, T]$ with values in H. In the sense of distributions, $\mathrm{d}^2u/\mathrm{d}t^2 + \mu = f$, which is precisely (3).

If v is an arbitrary element of $C^0([0, T]; H)$,

$$\lim_{\lambda \to 0} \phi_\lambda(v(t)) = \phi(v(t)).$$

Let v be such that $\phi(v) \in L^1(0, T)$. Then $\phi(v_\lambda)$ is also in $L^1(0, T)$. We have

(12)
$$\int_0^T (\phi_\lambda(v(t)) - \phi_\lambda(u_\lambda(t)))\,\mathrm{d}t \geqslant \int_0^T (\partial\phi_\lambda(u_\lambda(t)), v(t) - u_\lambda(t))\,\mathrm{d}t.$$

It is clear that the right-hand side tends to

$$\langle \mu, v - u \rangle$$

as λ goes to infinity.

We have

$$\lim \int_0^T \phi_\lambda(u_\lambda(t)) \geqslant \int_0^T \phi_\lambda(u_\lambda(t))\,\mathrm{d}t \leqslant \int_0^T \phi(u(t))\,\mathrm{d}t.$$

Hence, taking the upper limit in (12), we get

$$\int_0^T (\phi(v(t)) - \phi(u(t)))\,\mathrm{d}t \geqslant \langle \mu, v - u \rangle, \text{ i.e. (4)}$$

Interpretation of μ.

Let $\mathrm{d}\mu = g\,.\,\mathrm{d}t + \mathrm{d}\mu_s$ be the decomposition of μ with respect to the Lebesgue measure on $[0, T]$. Then Corollary 5.A of ref. [5], gives us the following results:

(13) $g(t) \in \partial\phi(u(t))$ almost everywhere (with respect to the Lebesgue measure) on $[0, T]$.

Let $N(t)$ be the normal cone to $K_0 = \overline{\mathrm{dom}\,\phi}$ at $u(t)$. Let $\mu_s = h\,.\,|\mu_s|$ with h a μ_s-integrable function. Then

(14) $h(t) \in N(t) \quad |\mu_s|$ − almost everywhere on $[0, T]$.

We shall often say in what follows that μ is the measure associated with the solution u of (P).

3. The properties of $\mathrm{d}u/\mathrm{d}t$

We know that the injection $M^1(0, T) \subsetneq H^{-1}(0, T)$ is compact. As μ_λ converges vaguely to μ in $M^1(0, T; H) \cong (M^1(0, T))^N$, μ_λ converges strongly to μ in $H^{-1}(0, T; H) \cong (H^{-1}(0, T))^N$, and therefore

$$\frac{\mathrm{d}u_\lambda}{\mathrm{d}t} \to \frac{\mathrm{d}u}{\mathrm{d}t} \text{ strongly in } L^2(0, T; H).$$

As $|\mathrm{d}u_\lambda/\mathrm{d}t| \leqslant E$, for all t and λ, $(\mathrm{d}u_\lambda/\mathrm{d}t) \to (\mathrm{d}u/\mathrm{d}t)$ in $L^p(0, T; H)$, for all $p \in [1, \infty)$. On the other hand, as $\mathrm{d}^2u/\mathrm{d}t^2$ is a measure, $\mathrm{d}u/\mathrm{d}t$ has right and left limits in every point of $(0, T)$, a right limit in 0, and a left limit in T, and we can write

$$\frac{\mathrm{d}^- u}{\mathrm{d}t} = u_1 + \int_0^T f(s)\,\mathrm{d}s - \mu([0, T))$$

$$\frac{\mathrm{d}^+ u}{\mathrm{d}t} = u_1 + \int_0^T f(s)\,\mathrm{d}s - \mu([0, T]).$$

4. Energy conservation

It is clear that

$$\lim_{\lambda \to 0} \int_0^T \frac{|u_\lambda - J_\lambda u_\lambda|^2}{\lambda}\,\mathrm{d}t = 0,$$

because $u_\lambda - J_\lambda u_\lambda \to 0$ in $C^0([0, T]; H)$ and $(u_\lambda - J_\lambda u_\lambda)/\lambda \rightharpoonup \mu$ in $M^1([0, T]; H)$. On the other hand, $(|u_\lambda - J_\lambda u_\lambda|^2)/\lambda$ is bounded in $L^\infty(0, T)$. Thus, $(|u_\lambda - J_\lambda u_\lambda|^2)/\lambda$ converges to 0 in $L^\infty(0, T)$ weak*. As $\phi_\lambda(u_\lambda)$ converges to a certain χ, $\phi(J_\lambda u_\lambda) - \phi_\lambda(u_\lambda) - |u_\lambda + J_\lambda u_\lambda|^2/(2\lambda)$ has the same limit in $L^\infty(0, T)$ weak*. We have the inequality

$$\int_0^T (\phi(u) - \phi(J_\lambda u_\lambda))\,\mathrm{d}t \geqslant \langle \mu_\lambda, u - J_\lambda u_\lambda \rangle \qquad \text{as } \mu_\lambda(t) \in \partial\phi(J_\lambda u_\lambda(t)) \text{ for } t \in [0, T].$$

Passing to the limit:

$$\int_0^T (\phi(u) - \chi)\,\mathrm{d}t \geqslant 0.$$

But:

$$\lim \phi(J_\lambda u_\lambda) \geqslant \phi(u).$$

This implies $\chi \geqslant \phi(u)$, thus proving that $\chi = \phi(u)$.

If $\mu(\{t_0\}) = 0$, we see from (9) that

$$\lim_{\lambda \to 0} \phi_\lambda(u_\lambda(t_0)) = \lim_{\lambda \to 0} \phi(u_\lambda(t_0)) = \tfrac{1}{2}|u_1|^2 + \phi(u_0) + \int_0^t (\dot{u}(s) f(s))\,\mathrm{d}s;$$

hence:

$$\tfrac{1}{2}|\dot{u}(t)|^2 + \phi(u(t)) = \tfrac{1}{2}|u_1|^2 + \phi(u_0) + \int_0^t (\dot{u}(s), f(s))\,\mathrm{d}s, \text{ almost everywhere on } [0, T].$$

5. Initial conditions

It is quite clear that $u(0) = u_0$.

To finish the proof, we need only consider the interpretation (13), (14) of μ. In fact, $N(t) = \partial\psi_{K_0}(u(t))$. If $\mathrm{d}^+ u/\mathrm{d}t(0) \neq u_1$, then $\mu_s = (-u_1 + \mathrm{d}^+ u/\mathrm{d}t(0))\,\delta_0 + \tilde{\mu}_s$, and the conclusion of Part (a) of Theorem 1 holds.

Part (b)

The case when $u_0 \in \operatorname{int} K_0$ is quite simple. So we make the hypothesis $u_0 \in \partial K_0$. The idea of the proof is to compare u_λ with the solution v_λ of

$$(15) \qquad \begin{cases} \dfrac{\mathrm{d}^2 v_\lambda}{\mathrm{d}t^2} + (\partial\psi_C)_\lambda (v_\lambda - u_0) = 0 \\ v_\lambda(0) = u_0 \\ \dfrac{\mathrm{d}v_\lambda}{\mathrm{d}t} = u_1 \end{cases}$$

Notice that $\partial\psi_{C\lambda}(x) = (x - P_C x)/\lambda$;
v_λ is given explicitly by

$$v_\lambda(t) = u_0 + tP_C u_1 + \sqrt{\lambda}\sin(t/\sqrt{\lambda})(u_1 - P_C u_1) \qquad \text{if } 0 \leqslant t \leqslant \pi\sqrt{\lambda}$$
$$v_\lambda(t) = u_0 + tP_C u_1 + (t - \pi\sqrt{\lambda})(u_1 - P_C u_1) \qquad \text{if } t \geqslant \pi\sqrt{\lambda}$$

We can check these formulas using the fact that the decomposition of an arbitrary element of H into the sum of its projections onto C and $C^\perp = N$ is unique. This result is given in ref. [3], Lemma 2.2. Knowing that $(\partial\psi_C)_\lambda$ is Lipschitzian, we can see that v_λ is the solution of (15). Let us show now that $\partial\phi_\lambda$ is locally near $(\partial\psi_c)_\lambda$ in an adequate sense.

PROPOSITION 1. If ϕ is Lipschitzian in a neighbourhood of u_0, there exists positive ρ and λ_0 such that, for $|x - u_0| \leqslant \rho$ and $\lambda \leqslant \lambda_0$:

$$(\partial\psi_c)_\lambda(x) - \partial\phi_\lambda(x) = h(x - u_0)/\lambda + A_\lambda(x) \tag{16}$$

where: $h(x - u_0) = P_{K_0}(x) - P_C(x - u_0)$;

$|h(x)| \leqslant |x|\varepsilon(|x|)$ and $\varepsilon(r)$ decreases to zero as r goes to zero;
$h(x) = 0$ on $K_0 - u_0$
$|A_\lambda(x)| \leqslant k$

Proof. 1. Let ϕ be Lipschitzian over $\operatorname{dom}\phi \cap \overline{B_\sigma(u_0)}$ with Lipschitz constant k. Then $\operatorname{dom}\phi \cap \overline{B_\sigma(u_0)}$ is closed: in fact, if $(x_n)_n$, is a sequence of points of $\operatorname{dom}\phi \cap \overline{B_\sigma(u_0)}$ converging to x_∞, we have

$$\phi(x) \leqslant \lim_{n\to\infty} \phi(x_n) \leqslant \lim_{n\to\infty} (k|x_n - x_p| + \phi(x_p)) = \phi(x_p) + |x - x_p|,$$

which shows that $x \in \operatorname{dom}\phi$.

2. Lipschitzian extension of $\phi|_{\overline{B_\sigma(u_0)}}$.
Denote by $\tilde{\phi}$ the convex function equal to ϕ in $\operatorname{dom}\phi \cup \overline{B_\sigma(u_0)}$ and $+\infty$ elsewhere. Obviously $\tilde{\phi}$ is lower semicontinuous.
Define a convex function χ by
$\chi(x) = \sup\{\tilde{\phi}(y) + (z, x - y) : y \in \operatorname{int}\operatorname{dom}\tilde{\phi} \text{ and } z \in \partial\tilde{\phi}(y)\}$.
In fact χ is a Lipschitzian extension of $\tilde{\phi}|_{\operatorname{dom}\tilde{\phi}}$: $\chi(x)$ is never infinite, thanks to the relation

$$\chi(x) \geqslant \tilde{\phi}(y) + k|x - y|$$

(clearly $\partial\tilde{\phi}$ is bounded by k on $\operatorname{int}\operatorname{dom}\tilde{\phi}$).
If $x \in \operatorname{dom}\tilde{\phi}$, $\chi(x) \geqslant \tilde{\phi}(x)$. On the other hand, given $\varepsilon > 0$, there exists a $y_\varepsilon \in \operatorname{int}\operatorname{dom}\tilde{\phi}$ and a $z_\varepsilon \in \partial\tilde{\phi}(y_\varepsilon)$ such that

$$\chi(x) \leqslant \tilde{\phi}(y_\varepsilon) + (z_\varepsilon, x - y_\varepsilon) + \varepsilon.$$

Therefore

$$\chi(x) \leqslant \tilde{\phi}(x) + \varepsilon.$$

As ε is arbitrary, we obtain $\chi(x) = \tilde{\phi}(x)$. We can extend this equality to all of the domain of $\tilde{\phi}$, by continuity. Let us show that χ is Lipschitzian: let x and x' be given in H; we have

$$\chi(x') - \chi(x) \leqslant \tilde{\phi}(y'_\varepsilon) + (z'_\varepsilon, x' - y'_\varepsilon) + \varepsilon - \tilde{\phi}(y'_\varepsilon, x - y'_\varepsilon)$$
$$= (z'_\varepsilon, x' - x) + \varepsilon \leqslant k|x' - x| + \varepsilon,$$

if ε is arbitrary and y'_ε and z'_ε are chosen adequately. We can bound from below $\chi(x') - \chi(x)$ by the same type of argument, and thus we have shown that $|\chi(x') - \chi(x)| \leqslant k|x' + x|$.

3. Decomposition of $\tilde{\phi}$.

Denote by ψ the indicator function of dom $\tilde{\phi} = \tilde{K}$.
Obviously $\tilde{\phi} = \psi + \chi$; then by Theorem 23.8 of ref. [2], $\partial\tilde{\phi}(x) = \partial\chi(x) + \partial\psi(x)$ for all x in $\tilde{K}$. We have

$$\partial\tilde{\phi}_\lambda(x) = (x - (I + \lambda\partial\tilde{\phi})^{-1} x)/\lambda.$$

Let $(I + \lambda\partial\tilde{\phi})^{-1}x = z_\lambda$:

$$z_\lambda + \lambda\partial\psi(z_\lambda) + \lambda g_\lambda \ni x$$

where $g_\lambda \in \partial\chi(z_\lambda)$ and is thus bounded by k;

$$z_\lambda = (I + \lambda\partial\psi)^{-1}(x - \lambda g_\lambda).$$

$$\begin{aligned} |\partial\tilde{\phi}_\lambda(x) - \partial\psi_\lambda(x)| &= |[x - (I + \lambda\partial\psi)^{-1}(x - \lambda g_\lambda) - x + (I + \lambda\partial\psi)^{-1}x]|/\lambda \\ &= 1/\lambda\,|(I + \lambda\partial\psi)^{-1}(x - \lambda g_\lambda) - (I + \lambda\partial\psi)^{-1}x| \leqslant |g_\lambda| \leqslant k. \end{aligned}$$

Lemma 4.6. of Zarantonello [3] allows us to write: $P_{\tilde{K}}(x + u_0) = u_0 + P_C x + h(x)$ with

$$|h(x)| \leqslant |x|\varepsilon(|x|) \qquad \text{and } h(x) = 0 \text{ if } x \in \tilde{K} - u_0.$$

Then

$$|\partial\psi_\lambda(x) - (\partial\psi_C)_\lambda(x - u_0)| = |P_C(x - u_0) - P_{\tilde{K}}(x - u_0)|/\lambda \leqslant \varepsilon(|x - u_0|)\,|x - u_0|.$$

4. Comparison of $\partial\phi_\lambda$ and $\partial\tilde{\phi}_\lambda$

The sequence $(I + \lambda\partial\phi)^{-1}$ converges to $P_{K_0}x$ when λ goes to zero; but $(I + \lambda\partial\phi)^{-1}$ is a contraction; this convergence is therefore uniform on compact sets. So, there exists a function j of ρ and λ, decreasing to zero as $\lambda \to 0$, such that $|(I + \lambda\partial\phi)^{-1}x - P_{K_0}x| \leqslant j(\lambda, \rho)$ on the ball $B_\rho(u_0)$, and in an analogous fashion

$$|(I + \lambda\partial\tilde{\phi})^{-1}x - P_{\tilde{K}}x| \leqslant \tilde{j}(\lambda, \rho) \qquad \text{on the ball } B_\rho(u_0).$$

$$\text{Clearly, } P_{K_0}(B_\rho(u_0)) \subset B_\rho(u_0) \qquad \forall\, \rho > 0$$
$$P_{\tilde{K}}(B_\rho(u_0)) \subset B_\rho(u_0) \qquad \forall\, \rho > 0.$$

Take $\rho = \sigma/2$, and choose λ_0 so small that

$$\max\,(j(\lambda_0, \sigma/2), \tilde{j}(\lambda_0, \sigma/2)) < \sigma/2.$$

Then $|(I + \lambda\partial\phi)^{-1}x - u_0| < \sigma$, and $|(I + \lambda\partial\tilde{\phi})^{-1}x - u_0| < \sigma$ if $|x| < \sigma/2$. Let $y = (I + \lambda\partial\phi)^{-1}x$ and $\tilde{y} = (I + \lambda\partial\tilde{\phi})^{-1}x$. We know that $\phi|_{B_\sigma(u_0)} = \tilde{\phi}|_{B_\sigma(u_0)}$.
Therefore

$$(1/\lambda)(x - y, v) + \phi(y + v) - \phi(y) \geqslant 0 \qquad \forall\, v \in H$$
$$(1/\lambda)(x - \tilde{y}, \tilde{v}) + \tilde{\phi}(\tilde{y} + \tilde{v}) - \tilde{\phi}(\tilde{y}) \geqslant 0 \qquad \forall\, \tilde{v} \in H.$$

If we add these two inequalities, taking $v = \tilde{y} - y$, and $\tilde{v} = -v$, we obtain $\tilde{y} = y$. This achieves the proof of Proposition 1.

LEMMA 1. u_λ and v_λ satisfy the inequalities:

$$|u_\lambda(t) - v_\lambda(t)| \leqslant ch\frac{t}{\sqrt{\lambda}}\left(k\frac{t^2}{2} + E\frac{t^3}{3!\lambda}\varepsilon(Et)\right) \tag{17}$$

$$|\dot{u}_\lambda(t) - \dot{v}_\lambda(t)| \leqslant kt + \frac{Et^2}{2\lambda}\varepsilon(Et) + \frac{1}{\lambda}ch\frac{t}{\sqrt{\lambda}}\left(k\frac{t^3}{3!} + \frac{Et^4}{4!\lambda}\varepsilon(Et)\right). \tag{18}$$

Proof. Define a transformation T_λ on $C^\circ([O, T]; H)$ by

$$(T_\lambda w)(t) = u_0 + tu_1 + \int_0^t (f(s) - \partial\phi_\lambda(w(s)))(t - s)\,ds.$$

Then u_λ satisfies

$$T_\lambda u_\lambda = u_\lambda.$$

Estimate $T_\lambda^n w - T_\lambda^n \hat{w}$ by a standard recurrence argument:

$$|(T_\lambda w - T_\lambda(\hat{w})(t)| \leqslant \frac{1}{\lambda}\frac{t^2}{2}\|w - \hat{w}\|_{C^0([0,t_0])} \qquad \text{if } t \leqslant t_0.$$

Then

$$|(T_\lambda^n w - T_\lambda^n \hat{w})(t)| \leqslant \frac{t^{2n}}{(2n)!\,\lambda^n}\|w - \hat{w}\|_{C^0([0,t_0])} \qquad \text{if } t \leqslant t_0$$

As we know that $T_\lambda^n w$ converges to u_λ as $n \to \infty$, for any initial w, and on any compact time interval, we may write

$$|u_\lambda(t) - w(t)| \leqslant \sum_{n \geqslant 0} \|T_\lambda^{n+1}w - T_\lambda^n w\|_{C^0([0,t_0])} \leqslant ch\frac{t}{\sqrt{\lambda}}\|T_\lambda w - w\|_{C^0([0,t_0])} \qquad \text{if } t \leqslant t_0.$$

Take $w = v_\lambda$ defined by (15).

$$(T_\lambda v_\lambda - v_\lambda)(t) = u_0 + tu_1 - \int_0^\lambda \partial\phi_\lambda(v_\lambda(s))(t - s)\,ds - u_0 - tu_1 + \int_0^t (\partial\psi_C)_\lambda(v_\lambda(s))(t - s)\,ds.$$

If we assume $t_0 \leqslant \rho/E$ and $\lambda \leqslant \lambda_0$, we may apply the conclusions of Proposition 1 for $x = v_\lambda(t)$. Therefore

$$|(T_\lambda v_\lambda - v_\lambda)(t)| \leqslant \int_0^t \left(k + \frac{Es\varepsilon(Es)}{\lambda}\right)(t - s)\,ds \leqslant k\frac{t_0^2}{2!} + E\frac{t_0^3}{3!\lambda}\varepsilon(Et_0) \qquad \text{if } t \leqslant t_0,$$

and finally we obtain the estimate (17).
The estimate (18) is a straightforward consequence of (17), thanks to the relation

$$|\dot{u}_\lambda(t) - \dot{v}_\lambda(t)| \leqslant \int_0^\lambda |(\partial\psi_c)_\lambda(v_\lambda(t)) - \partial\phi_\lambda(v_\lambda(t)|\,dt + \int_0^\lambda |\partial\phi_\lambda(v_\lambda(t)) - \partial\phi_\lambda(u_\lambda(t))|\,dt.$$

End of proof of Theorem 1, part b.

Denote $\tilde{u}_1 = 2P_C u_1 - u_1$. Clearly, $\tilde{u}_1$ is an element of C. Let us notice that, for any given $\eta > 0$, there exists a $\delta(\eta) > 0$ such that, if $|P_{K_0}v - u_0| \leqslant \delta(\eta)$, then

$$(v - P_{K_0}v, \tilde{u}_1) \leqslant \eta|\tilde{u}_1|\,|v - P_{K_0}v|.$$

If it were not the case, we could find a sequence v_n with Pv_n converging to u_0, and a strictly positive number η_0 such that

$$(v_n - P_{K_0}v_n, \tilde{u}_1) \geqslant \eta_0 |\tilde{u}_1| \, |v_n - P_{K_0}v_n|.$$

We can see that $(v_n + Pv_n)/|v_n + Pv_n|$ converges to a certain w, which must be in N, and therefore, we obtain a contradiction. We may assume $\eta(\delta)$ to be an increasing function of δ, such that

$$|P_{K_0}v - u_0| \leqslant \delta \text{ implies } (v - P_{K_0}v, \tilde{u}_1) \leqslant \eta(\delta)\,|\tilde{u}_1|$$

Write now

$$\begin{aligned}
(u_\lambda(t), \tilde{u}_1) &= (u_\lambda(\pi\sqrt{\lambda}), \tilde{u}_1) + (\dot{u}_\lambda(\pi\sqrt{\lambda}), \tilde{u}_1)(t - \pi\sqrt{\lambda}) + \left(\int_{\pi\sqrt{\lambda}}^t (f(s) - A_\lambda u_\lambda(s))(t-s)\,\mathrm{d}s, \tilde{u}_1 \right) \\
&\quad - \int_{\pi\sqrt{\lambda}}^t ([u_\lambda(s) - P_{K_0}u_\lambda(s)], \tilde{u}_1)(t-s)\,\mathrm{d}s \\
&\geqslant (u_\lambda(\pi\sqrt{\lambda}), \tilde{u}_1) + (\dot{u}_\lambda(\pi\sqrt{\lambda}), \tilde{u}_1)(t + \pi\sqrt{\lambda}) \\
&\quad + |f|_{L^2}\frac{t^{3/2}}{\sqrt{\lambda})} + k\frac{t^2}{2} - \eta(Et)\int_{\pi\sqrt{\lambda}}^t (1/\lambda)|u_\lambda(s) - P_{K_0}u_\lambda(s)|\,|\tilde{u}_1|\,(t-s)\,\mathrm{d}s
\end{aligned}$$

From Proposition 1, and the estimate (11), we have:

$$\int_0^T (1/\lambda)|u_\lambda(s) - P_{K_0}u_\lambda(s)|\,\mathrm{d}s \leqslant C' + kT.$$

We obtain, in the limit as $\lambda \to 0$

$$(u(t) - u_0, \tilde{u}_1) \geqslant |\tilde{u}_1|^2 t - o(t), \text{ from where}$$

$$\left(\frac{\mathrm{d}^+u}{\mathrm{d}t}(0), \tilde{u}_1\right) \geqslant |\tilde{u}_1|^2.$$

On the other hand, as a result of (6), and of the fact that ϕ is continuous in a neighbourhood of u_0

$$\left|\frac{\mathrm{d}^+u}{\mathrm{d}t}(0)\right| = |u_1| = |\tilde{u}_1|. \quad \text{Conclusion (8) is now clear.}$$

This completes the proof of Theorem 1.

3. COUNTEREXAMPLES TO WELLPOSEDNESS

3.a *Discontinuity*

The idea of this counterexample is quite simple.

Take $H = \mathbf{R}^2$, $K = \{(x_1, x_2): x_1 \geqslant 0,\ x_1 + x_2 \geqslant 0\}$, $\phi = \psi_k$, $f = 0$; take as initial data $u_0^h = (1 - h, \frac{1}{2} + h)$, $u_1^h = (-1, -\frac{1}{2})$ where $|h| < \frac{1}{2}$. It is easy to check that the solutions are unique.

If $h = 0$, according to part (b) of Theorem 1, the solution is

$$\begin{aligned}
u^0(t) &= (1 - t, \tfrac{1}{2}(1 - t)) \qquad &&\text{if } t \leqslant 1 \\
u^0(t) &= (\tfrac{1}{2}(t - 1), t - 1) \qquad &&\text{if } t > 1.
\end{aligned}$$

If $h > 0$, the solution is

$$u^h(t) = (1 - h - t, \tfrac{1}{2} + h - \tfrac{1}{2}t) \qquad \text{if } t \leqslant 1$$

and it has a unique reflection at time $t = 1$ on the side $x_1 + x_2 = 0$ of K. Then

$$u^h(t) = (\tfrac{1}{2}(t - 1) - h, t - 1 + h) \qquad \text{if } t > 1.$$

If $h < 0$, the solution has a first reflection at time $t = 1 + 2h$; then $u^h(t) = (-3h - (t - 1 - 2h), (t - 1 - 2h)/2)$ until the time $1 + 4h$, when it has a second reflection, after which

$$u^h(t) = (-\tfrac{1}{2}t + \tfrac{1}{2} - h, t - 1 + h).$$

As h tends to zero, remaining negative, u^h tends to $\tilde{u}^0$ defined by

$$\tilde{u}^0(t) = (1 + t, \tfrac{1}{2}(1 - t)) \qquad \text{if } t \leqslant 1$$

$$\tilde{u}_0(t) = (-\tfrac{1}{2}(t - 1), t - 1) \qquad \text{if } t > 1.$$

We can see now that, as a function of h, u^h is right- but not left-continuous.

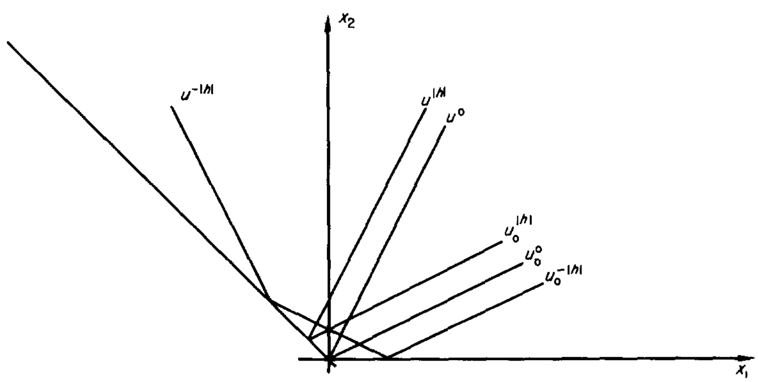

Fig. 2.

3. b. *Nonuniqueness*

Two examples will be given, both of them with $\phi = \psi_K$.

(i). Take $H = \mathbf{R}$, $K = \mathbf{R}^+$, $\phi = \psi_K$, $u_0 = u_1 = 0$.

We seek an infinitely differentiable nonpositive f which gives us two solutions. The first solution will be formed of an infinite number of arches, smaller and smaller as t gets nearer to 0 (see figure below). The associated measure μ will then be atomic. The second solution will be $u = 0$, $\mu = f$.

More precisely, let $\rho \in \mathscr{D}(\mathbf{R})$ be an even, nonnegative function, the support of which is included in $[-1, +1]$, and satisfying $\int_R \rho(x)\,dx = 1$. Denote $\rho_\varepsilon(x) = \rho(x/\varepsilon)$. Let $y_\varepsilon = 1_{[\varepsilon/2, 1 - \varepsilon/2]} * \rho_{\varepsilon/2}$ with $\varepsilon < \frac{1}{2}$. Clearly $y_\varepsilon = 1$ on $[\varepsilon, 1 - \varepsilon]$, supp $y_\varepsilon \subset [0, 1]$, and y_ε is C^∞. It is an easy exercise to check that

$$2\int_0^1 y_\varepsilon(s)(1 - s)\,ds - \int_0^1 y_\varepsilon(s)\,ds = 0.$$

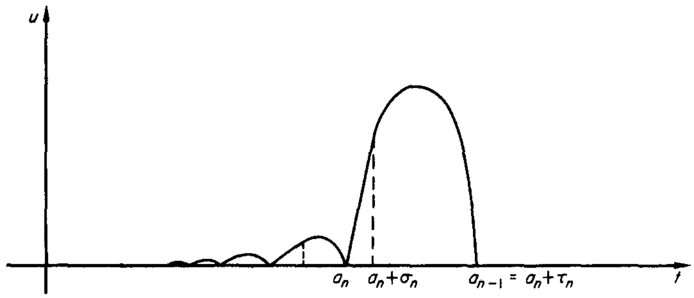

Fig. 3.

From now on we fix an ε, and write $y_\varepsilon = y$, $y_1 = \int_0^1 y(s)\,\mathrm{d}s$ and $y_2 = \int_0^1 y(s)(1+s)\,\mathrm{d}s$. Obviously $y_1 = 2y_2$. We look for an f defined on each interval $[a_n, a_{n-1}]$ as follows: $f = 0$ on $[a_n, a_n + \sigma_n]$

$$f(t) = -\eta_n y\left(\frac{t - a_n - \sigma_n}{\tau_n - \sigma_n}\right) \text{ on } [a_n + \sigma_n, a_n + \tau_n] = [a_n + \sigma_n, a_{n-1}].$$

f is C^∞ except perhaps in 0, if it is defined.

We have necessarily

$$\ddot{u} = f \qquad \text{on } [a_n, a_{n-1}]$$

$$u(a_n) = u(a_{n-1}) = 0$$

If we write $\dfrac{\mathrm{d}^+u}{\mathrm{d}t}(a_n) = v_n$ then $\dfrac{\mathrm{d}^-u}{\mathrm{d}t}(a_{n-1}) = -v_{n-1}$.

Our aim is now to look for necessary conditions which will insure that f has a meaning and is C^∞, and u has a meaning and is a solution of (P).

$$u(t) = v_n(t - a_n) \qquad\qquad \text{on } [a_n, a_n + \sigma_n]$$

$$u(t) = v_n(t - a_n) - \eta_n \int_{a_n+\sigma_n}^{t} y(s)(t+s)\,\mathrm{d}s \quad \text{on } [a_n + \sigma_n, a_{n-1}],$$

which gives us

$$v_n\tau_n = \eta_n y_2(\tau_n - \sigma_n)^2$$

$$v_n + v_{n-1} = \eta_n y_1(\tau_n - \sigma_n).$$

Let $v_{n-1} = \alpha_n v_n$. Then

$$\tau_n = \sigma_n \frac{\alpha_n + 1}{\alpha_n - 1} \quad \text{and } \eta_n\sigma_n = \frac{v_n(\alpha_n^2 - 1)}{2y_1}.$$

Choose $\alpha_n = n$ for $n \geqslant 4$. This implies

$$\eta_n\sigma_n = \frac{v_4}{2y_1}\frac{(n^2-1)\,4!}{n!} = \frac{12v_4}{y_1}\frac{n^2-1}{n!}.$$

Take

$$\eta_n = 12\frac{v_4}{y_1}\frac{1}{(n-4)!} \quad \text{and } \sigma_n = \frac{n+1}{n(n-2)(n-3)}.$$

For u to be a solution of (P), we have only to check that

$$\sum_{n\geq 4} \mu(\{a_n\}) = 2\sum_{n\geq 4} v_n = 2\sum_{n\geq 4}\frac{1}{n!}\,24\,v_4 < +\infty$$

$$\sum_{n\geq 4}\tau_n = \sum_{n\geq 4}\frac{n+1}{n(n-2)(n-3)}\frac{n+1}{n-1} < +\infty.$$

Then $a_4 < +\infty$ and we can choose $T = a_4$.

For f to be C^∞ up to zero we need

$$\eta_n/\Big(\sum_{m\geq n-1}\tau_n\Big)^r \sim \frac{1}{(n-4)!}(C_n)^p\frac{v_1}{2y_1} < +\infty \quad \forall p;$$

thus, we obtain the supplementary information that $f^{(p)}(0) = 0, \forall p \in \mathbf{N}$. Clearly 0 with associated measure $\hat{\mu} = f$ is a solution. Thus we have built an infinitely differentiable f such that (P) has two solutions.

(ii) Take $H = \mathbf{R}^2, f = 0$. We intend to build a convex set K of C^∞ boundary, such that with an initial velocity tangent to the boundary, we obtain two different solutions; one "along" the boundary, the other having an infinity of reflexions on it.

We shall use the results and notations of the preceding paragraph. Let us define a continuous parametrized curve with values in $\mathbf{C} \simeq \mathbf{R}^2$ as follows:

$$F(t) = F(a_n) + e^{i\beta_n}(t-a_n) + e^{i(\beta_n - \pi/2)}u(t) \qquad \text{if } t\in[a_n, a_{n-1}]\ 0 \leq t \leq T = a_4.$$

This curve will be continuously differentiable if

$$e^{i\beta_n}\left(1 - i\frac{d^+u}{dt}(a_n)\right) = e^{i\beta_{n+1}}\left(1 - i\frac{d^-u}{dt}(a_n)\right).$$

Then, necessarily $\beta_n - \beta_{n+1} = 2\arctan v_n$. As $v_n = 24v_4/n!$, the series $\sum\beta_n$ converges. Moreover, if we choose v_4 small enough, $\sum_{n=4}\beta_n < \langle\pi/2$, which means that $t \mapsto {}^nF(t)$ will have no tangent parallel to the imaginary axis. We have $F''(t) = e^{i(\beta_n - \pi/2)}f(t)$ on $[a_n, a_{n-1}]$. But f is constructed in such a way that $f^{(p)}(a_n) = 0$ for all $p \geq 0$, $n \geq 4$. Therefore F is C^∞ on $[0, T]$.^{p}Moreover, the sign of the curvature of $F([0, T])$ is the sign of $\dot{F}_1\ddot{F}_2 - \dot{F}_2\ddot{F}_1$, where $F = F_1 + iF_2$.

We can easily check that $\dot{F}_1\ddot{F}_2 - \dot{F}_2\ddot{F}_1 = -f(t) \geq 0$. We can thus see that $F([0, T])$ is the boundary of a convex set K of $\mathbf{C} \simeq \mathbf{R}^2$.

Take now initial conditions $u_0 = 0$, $\dot{u}_0 = a > 0$, and define a function ψ_a on $[0, T]$ by

$$\psi_a(t) = \frac{1}{a}\int_0^t |\dot{F}(\sigma)|\,d\sigma.$$

Let ϕ_a be the reciprocal of ψ_a.

Then $v(t) = F(\phi_a(t))$ is a solution of (P) on $[0, \psi_a(T)]$.

In fact, $v(t) \in K$, for all $t \in [0, \psi_a(T)]$.

$$\frac{\mathrm{d}v}{\mathrm{d}t} = a\,\dot{F}(\phi_a(t))/|\dot{F}(\phi_a(t))|$$

$$\frac{\mathrm{d}^2 v}{\mathrm{d}t^2} = a\frac{\mathrm{d}}{\mathrm{d}t}\,[\dot{F}(\phi_a(t))/|\dot{F}(\phi_a(t))|].$$

As $\dot{F}(\sigma) \neq 0$, for all $\sigma \in [0, T]$, $\mathrm{d}^2u/\mathrm{d}t^2$ is a bounded function which can be identified with a bounded measure. On the other hand, it is clear that $\mathrm{d}^2v/\mathrm{d}t^2$ is normal to ∂K, $(\mathrm{d}v/\mathrm{d}t)$ is continuous at every point, $|\mathrm{d}v/\mathrm{d}t| = a$ and the initial conditions are satisfied. Therefore u is a solution of (P) in the sense of Definition 1.

The other solution will be given by

$$\tilde{v}(t) = F(a_n) + \mathrm{e}^{i\beta_n}(at - a_n) \quad \text{on} \left[\frac{a_n}{a}, \frac{a_{n-1}}{a}\right]$$

Clearly, $\tilde{v}(t) \in K$ for all t. We have

$$\frac{\mathrm{d}^+\tilde{v}}{\mathrm{d}t}(a_n/a) = a\,\mathrm{e}^{i\beta_n}$$

$$\frac{\mathrm{d}^-\tilde{v}}{\mathrm{d}t}(a_n/a) = a\,\mathrm{e}^{i\beta_{n+1}}$$

$$\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}t^2} = 0 \quad \text{if } t \in (a_n, a_{n-1}).$$

Therefore

$$\frac{\mathrm{d}^2\tilde{v}}{\mathrm{d}t^2} = a\sum_{n\geqslant 1} \delta_{a_n/a}(\mathrm{e}^{i\beta_n} - \mathrm{e}^{i\beta_{n+1}}) = -\,\tilde{\mu}.$$

It is easy to check, using the interpretation of the measure associated with a solution of (P), (13), (14), that $\tilde{v}$ is a solution of (P); clearly, the energy is conserved, and the initial conditions are satisfied.

Comments on this part. These results are closely connected with results on the propagation of singularities of a hyperbolic problem on a manifold with boundary. The first one obviously refers to diffraction in an angle. The second one was figured out by thinking of a tennis player who succeeds in making his ball bounce higher and higher, from a rest position, only by hitting it downwards. Professor L. Amerio told me that he had built an analogous example. The third type of counterexample is given, with a different construction by M. E. Taylor in his paper [4] concerning reflection of singularities of wave equations in an exterior domain of $\mathbf{R}^N$. When the complement of this domain is not convex, Taylor's theorem fails as there appear phenomena of nonuniqueness of the wave front set. I cannot see, presently, how all these results could be taken in account together in reasonable mathematical theory.

4. A UNIQUENESS THEOREM IN A PARTICULAR CASE

THEOREM 2. Let $\phi = \psi_K, f = 0$. Assume that the boundary ∂K of the closed convex set K is of class C^3, and that its gaussian curvature is strictly positive. Then the problem (P) admits a unique solution on $[0, +\infty]$ in the sense of Definition 1. Moreover, if the initial data u_0 is given on ∂K and u_1 is tangent to ∂K, then u runs along the geodesic of ∂K passing through u_0 and tangent to u_1 with the speed $|u_1|$. If the initial data are not such, then u is never tangent to ∂K, and it has a finite number of reflections in a finite time.

Proof. Denote by $\mathscr{S}_K(u)$ the tangent cone to K at u, i.e. $\mathscr{T}_K(u) = \overline{\bigcup_{\tau > 0} \tau(K - u)}$ and by $n(u)$ the exterior unit normal to K at u. Remark that

$$\mathscr{S}_K(u) = \{v : (v, n(u)) \geqslant 0\} \qquad \text{for all } u \in \partial K.$$

Let us first prove the local uniqueness at every point t. If $u(t) \in \operatorname{int} K$, or if $u(t) \in \partial K$ and $(\mathrm{d}^- u/\mathrm{d}t)(t)$ is the interior of $\mathscr{S}_K(u(t))$, the local uniqueness is immediate.

If $u(t) \in \partial K$, and $\mathrm{d}^- u/\mathrm{d}t\,(t) \notin \mathscr{S}_K(u(t))$, then

$$\left|\frac{\mathrm{d}^+ u}{\mathrm{d}t}(t)\right| = \left|\frac{\mathrm{d}^- u}{\mathrm{d}t}(t)\right|.$$

and

$$\frac{\mathrm{d}^+ u}{\mathrm{d}t}(t) - \frac{\mathrm{d}^- u}{\mathrm{d}t}(t) = kn(u(t))$$

according to Theorem 1.

Necessarily the vector

$$\frac{\mathrm{d}^+ u}{\mathrm{d}t}(t) = \frac{\mathrm{d}^- u}{\mathrm{d}t}(t) - 2\left(\frac{\mathrm{d}^- u}{\mathrm{d}t}(t), n(u(t))\right) n(u(t))$$

is in the interior of $\mathscr{S}_K(u(t))$. The local uniqueness is then clear. The only difficult case is when $u(t) \in \partial K$, and $((\mathrm{d}^- u/\mathrm{d}t\,(t)), n(u(t))) = 0$. We need the following result.

LEMMA 2. If ∂K is of class C^3 and its gaussian curvature is strictly positive, then there exists no sequence of points of ∂K, $(u_m)_{m \in N}$ converging to u_∞, such that u_m is a reflexion point for all m, i.e.

$$\begin{cases} \dfrac{(u_m - u_{m-1}, n(u_m))}{|u_m - u_{m-1}|} = \dfrac{(u_m - u_{m+1}, n(u_m))}{|u_m - u_{m+1}|} \quad \text{and} \\[2ex] \dfrac{u_m - u_{m-1}}{|u_m - u_{m-1}} - \dfrac{(u_m - u_{m-1}, n(u_m))\, n(u_m)}{|u_m - u_{m-1}{}^1} = \dfrac{u_{m+1} - u_m}{|u_{m+1} - u_m|} - \dfrac{(u_{m+1} - u_m, n(u_m))\, n(u_m)}{|u_{m+1} - u_m|} \end{cases}$$

such that $u_{m+1} - u_m$ has a limit direction, and such that

$$\sum_{m \in \mathrm{N}} |u_{m+1} - u_m| < +\infty.$$

Let us show that this lemma implies local uniqueness. With $u_0 \in \partial K$, $(n(u(t_0)), (\mathrm{d}^- u/\mathrm{d}t)(t_0)) = 0$, then necessarily $((d^+ u/\mathrm{d}t)(t_0), n(u(t_0))) = 0$. Suppose that there is no right neighbourhood of t_0 such that $u(t)$ belongs to ∂K in all this neighbourhood. Then we can find a t_1, arbitrarily near t_0, such that $u(t_1) \notin \partial K$. There must exist at most a finite number of reflections between t_0 and t_1, in order to satisfy Lemma 2. Then $((\mathrm{d}^+ u/\mathrm{d}t(t_0)), n(u(t_0))) < 0$ and we obtain a contradiction. Therefore there exists an $\eta > 0$ such that $u(t) \in \partial K$ for $t_0 \leqslant t \leqslant t_0 + \eta$.

Let μ be the measure associated with u, and define a real valued measure (μ, z) with $z \in C^0([0, T]; H)$ by

$$\langle(\mu, z), \phi\rangle = \langle\mu, z\,\phi\rangle \qquad \forall\, \phi \in C^0([0, T]).$$

Let $s = (w - (w, n(u))n(u))\,\chi$ with $w \in H$, and $\chi \in C^0([0, T])$, $\operatorname{supp}\chi \subset [t_0, t_0 + \eta]$. It is clear that $s(t) \in \mathscr{S}_K(u(t))$, and s is universally integrable on $[0, T]$. According to the interpretation of μ, (13) and (14), necessarily $\langle\mu, s\rangle = 0$.

So

$$(\mu, w)|_{(t_0, t_0 + \eta)} = (n(u), w)(\mu|_{(t_0, t_0 + \eta)}, n(u)).$$

If we set $v = (\mu|_{(t_0, t_0 + \eta)}, n(u))$, we may then identify $\mu|_{(t_0, t_0 + \eta)}$ and $n(u)v$. Suppose $u(t_0) = 0$, which does not restrict the generality, and denote $n(t_0) = n_0$, $u_N = -(u, n_0)$, $u' = u - (u, n_0)n_0$. Represent ∂K in a neighbourhood of 0 as follows:

$$|u| \leqslant \alpha \quad \text{and } u_N = f(u') \quad \text{imply} \quad u \in \partial K.$$

Here f is convex, twice continuously differentiable, and $Df(0) = 0$. We differentiate the relation $u_N(t) = f(u'(t))$ on $[t_0, t_0 + \eta]$.

$$\frac{\mathrm{d}u_N}{\mathrm{d}t} = Df(u'(t))\frac{\mathrm{d}u'}{\mathrm{d}t}; \tag{19}$$

$\dfrac{\mathrm{d}u}{\mathrm{d}t}$ is continuous, as $\left|\dfrac{\mathrm{d}^+u}{\mathrm{d}t}\right| = \left|\dfrac{\mathrm{d}^-u}{\mathrm{d}t}\right|$, $\dfrac{\mathrm{d}^+u}{\mathrm{d}t} - \dfrac{\mathrm{d}^-u}{\mathrm{d}t} = kn(u)$ (for a certain real k), and

$$\left(\frac{\mathrm{d}^+u}{\mathrm{d}t}, n(u)\right) = \left(\frac{\mathrm{d}^-u}{\mathrm{d}t}, n(u)\right) = 0.$$

We now differentiate (19) in the sense of distributions:

$$\frac{\mathrm{d}^2u_N}{\mathrm{d}t^2} = D^2f(u'(t))\frac{\mathrm{d}u'}{\mathrm{d}t}\frac{\mathrm{d}u'}{\mathrm{d}t} + Df(u'(t))\frac{\mathrm{d}^2u'}{\mathrm{d}t}. \tag{20}$$

Explicitly

$$n(u) = \frac{(Df(u'); -1)}{(1 + |Df(u')|^2)^{1/2}},$$

whence

$$\frac{\mathrm{d}^2u'}{\mathrm{d}t^2} = \frac{-v\,Df(u')}{(1 + |Df(u')|^2)^{1/2}},$$

$$\frac{\mathrm{d}^2u_N}{\mathrm{d}t^2} = \frac{v}{(1 + |Df(u')|^2)^{1/2}},$$

$$\left(Df(u'), \frac{\mathrm{d}^2u'}{\mathrm{d}t^2}\right) = -\frac{v\,|Df(u')|^2}{(1 + |Df(u')|^2)^{1/2}}.$$

Substituting in relation (20) we obtain

$$v(1 + |Df(u')|^2)^{1/2} = D^2f(u')\frac{\mathrm{d}u'}{\mathrm{d}t}\frac{\mathrm{d}u'}{\mathrm{d}t}.$$

Therefore, v can be identified with a function and

$$\frac{d^2u}{dt^2} + \frac{D^2 f(u')\dfrac{du'}{dt}\dfrac{du'}{dt}}{(1 + |Df(u')|^2)^{1/2}} n(u) = 0.$$

This is precisely the equation of the geodesics of ∂K. As ∂K is of class C^3, we know there is uniqueness. By a classical argument, local uniqueness implies global uniqueness.

To complete the proof of Theorem 2, we need only to establish Lemma 2.

Proof of Lemma 2. Suppose there exists a sequence $u_m \in \partial K$ such that

(i) $u_m \to u_\infty$ as $m \to \infty$,

(ii) $\dfrac{Q_m(u_m - u_{m+1})}{|u_m - u_{m+1}|} = \dfrac{Q_m(u_m - u_{m-1})}{|u_{m-1} - u_m|}$,

(iii) $\dfrac{P_m(u_m - u_{m+1})}{|u_m - u_{m+1}|} = \dfrac{P_m(u_{m-1} - u_m)}{|u_{m-1} - u_m|}$

where $Q_m x = -(x, n(u_m))$, $P_m x = x + n(u_m) Q_m x$.

In a neighbourhood U of u_∞ we have the representations of ∂K given by

$$u \in \partial K \cap U \Rightarrow Q_m(u - u_m) = f_m(P_m(u - u_m)),$$

where f_m is of class C^3 and $Df_m(0) = 0$; f_m is convex.
Obviously

$$\frac{f_m(P_m(u_{m-1} - u_m))}{|P_m(u_{m-1} - u_m)|} = \frac{f_m(P_m(u_{m+1} - u_m))}{|P_m(u_{m+1} - u_m)|}. \tag{21}$$

On the other hand,

$$f_m(u') = D^2 f_m(0)\, u'u' + |u'|^2 g_m(u') \quad \text{if } (u', n(u_m)) = 0, \tag{22}$$

with

$$|g_m(u')| \leqslant h(|u'|).$$

The function h does not depend on m; $\lim_{r \to 0} h(r) = 0$, and h is increasing

$$\begin{aligned} \text{Set } \kappa_m &= D^2 f_m(0)\left(\frac{P_m(u_{m-1} - u_m)}{|P_m(u_{m-1} - u_m)|}\right)\left(\frac{P_m(u_{m-1} - u_m)}{|P_m(u_{m-1} - u_m|}\right) \\ &= D^2 f_m(0)\left(\frac{P_m(u_{m+1} - u_m)}{|P_m(u_{m+1} - u_m)|}\right)\left(\frac{P_m(u_{m+1} - u_m)}{|P_m(u_{m+1} - u_m)|}\right). \end{aligned}$$

From (21) and (22) we obtain:

$$\begin{aligned} \kappa_m(|P_m(u_{m-1} - u_m)| - |P_m(u_{m+1} - u_m)|) + |P_m(u_{m-1} - u_m)|\, g_m(P_m(u_{m-1} - u_m)) \\ - |P_m(u_{m+1} - u_m)|\, g_m(P_m(u_{m+1} - u_m)) = 0 \end{aligned} \tag{23}$$

Since we have assumed that the gaussian curvature of ∂K at u_∞ is strictly positive, $\kappa_m \geqslant \kappa_0 > 0$ for m large enough. Dividing both sides of (23) by $|P_m(u_{m-1} - u_m)|$ we obtain

$$\kappa_0\left|\frac{|P_m(u_{m+1}-u_m)|}{|P_m(u_{m-1}-u_m)|}-1\right| \leqslant \left|\frac{|P_m(u_{m+1}-u_m)|}{|P_m(u_{m-1}-u_m)|}-1\right| h(|P_m(u_{m+1}-u_m)|)$$
$$+ h(|P_m(u_{m-1}-u_m)|) + h(|P_m(u_{m+1}-u_m)|).$$

If we choose m_0 such that

$$m \geqslant m_0 \Rightarrow h(|P_m(u_{m+1}-u_m)|) \leqslant \kappa_0/2,$$

then

$$\frac{\kappa_0}{2}\left|\frac{|P_m(u_{m+1}-u_m)|}{|P_m(u_{m-1}-u_m)|}-1\right| \leqslant h(|P_m(u_{m-1}-u_m)|) + h(|P_m(u_{m+1}-u_m)|).$$

Set $l_m = |u_{m-1}-u_m|$. Finally

$$\frac{\kappa_0}{2}\left|\frac{|P_m(u_{m+1}-u_m)|}{|P_m(u_{m-1}-u_m)|}-1\right| \leqslant h(l_m) + h(l_{m+1}). \tag{24}$$

On the other hand we have

$$P_m(u_{m-1}-u_m) - f_m(P_m(u_{m-1}-u_m))\, n(u_m) = u_{m-1}-u_m.$$

Therefore

$$\big|l_m - |P_m(u_{m-1}-u_m)|\big| \leqslant |D^2 f_m(0)(P_m(u_{m-1}-u_m))(P_m(u_{m-1}+u_m))| +$$
$$+ |P_m(u_{m-1}-u_m)|^2\, h(|P_m(u_{m-1}-u_m)|).$$

So we have a constant K such that

$$\left|\frac{l_m}{|P_m(u_{m-1}-u_m)|}-1\right| \leqslant K\, l_m \tag{25}$$

and analogously

$$\left|\frac{l_{m+1}}{|P_m(u_{m+1}-u_m)|}-1\right| \leqslant K\, l_{m+1}. \tag{26}$$

From (24), (25) and (26) we obtain

$$|l_{m+1}/l_m - 1| \leqslant K'(h(l_m) + h(l_{m+1}) + l_m + l_{m+1}).$$

Set $\tilde{h}(r) = K'(h(r) + r)$, then

$$l_{m+1} \geqslant l_m + l_m(\tilde{h}(l_{m+1}) + \tilde{h}(l_m)).$$

Since l_m converges to 0, we can find an arbitrarily large m_0 such that $l_m \leqslant l_{m_0}$, for all $m \geqslant m_0$. Then

$$l_{m+1} \geqslant l_m - 2l_m\tilde{h}(l_{m_0}) \qquad \text{if } m \geqslant m_0.$$

$$\sum_{m\geqslant m_0+1} l_m \geqslant \sum_{m\geqslant m_0} l_m(1-2\tilde{h}(l_{m_0})) = \sum_{m\geqslant m_0+1}(1-2\tilde{h}(l_{m_0}))\, l_m + (1-2\tilde{h}(l_{m_0}))\, l_{m_0}.$$

Therefore

$$\sum_{m\geqslant m_0+1} l_m \geqslant \frac{1-2\tilde{h}(l_{m_0})}{2\tilde{h}(l_{m_0})}\cdot l_{m_0}.$$

It is clear that we can choose m_0 so large that the right hand side of this inequality is arbitrarily large. We obtain therefore a contradiction, as $u_{m+1} - u_m$ is supposed to have a limit direction when m tends to infinity. This completes the proof of Theorem 2.

The main results of this paper have been stated in ref. [6].

REFERENCES

1. BREZIS H., *Opérateurs Maximaux Monotones et Semi-Groupes de Contraction dans les Espaces de Hilbert.* North Holland (1973).
2. ROCKAFELLAR R. T., *Convex Analysis.* Princeton Univ. Press (1970).
3. ZARANTONELLO E. H., *Contributions to Nonlinear Functional Analysis.* pp. 237–424. Academic Press (1971).
4. TAYLOR M. E., *Communs pure appl. Math.* **38**, 1–38 (1976).
5. ROCKAFELLAR R. T., *Pacif. J. Math.* **39**, 439–469 (1971).
6. SCHATZMAN M., *C.r. Acad. Sci. Paris* **284**, série A, 603–606 (1977).